Published by Institute of Physics Publishing for SISSA/ISAS
Received: *February 12, 2003*
Accepted: *March 11, 2003*

Poincare covariant mechanics on noncommutative space.

Alexei A. Deriglazov*

Departamento de Matematica, Universidade Federal de Juiz de Fora
Juiz de Fora, MG, Brasil
E-mail: alexei@ice.ufjf.br

Abstract: The Dirac approach to constrained systems can be adapted to construct relativistic invariant theories on a noncommutative (NC) space. As an example, we propose and discuss relativistic invariant NC particle coupled to electromagnetic (EM) field by means of the standard term $A_\mu \dot{x}^\mu$. Poincare invariance implies deformation of the free particle NC algebra in the interaction theory. The corresponding corrections survive in the nonrelativistic limit.

Keywords: Space-Time Symmetries, Gauge Symmetry, Non-Commutative Geometry.

*On leave of absence from Dept. Math. Phys., Tomsk Polytechnical University, Tomsk, Russia.

JHEP03(2003)021

Contents

1. Introduction: noncommutative geometry and relativistic invariance

Noncommutativity of space coordinates [1, 2] can be naturally incorporated into the quantum field theory framework, at least for mechanical models [3]–[8], using the Dirac approach to constrained systems [9]–[12] appropriately adopted to the case. The method implies that one works with the lagrangian action in the first order form, the latter being characterized by doubling of the configuration space variables: $q^A \longrightarrow (q^A, v_A)$. Noncommutative (NC) version of an arbitrary nondegenerate mechanics can be obtained [7] by addition of a Chern-Simons type term for v_A to the first order action of the initial system: $S_{NC} = S_1(q, v) + \int d\tau \dot{v}_A \theta^{AB} v_B$. Nontrivial bracket for the position variables q^A appears in this case as the Dirac bracket, after taking into account the constraints present in the model. The numerical matrix $\theta^{AB} = -\theta^{BA}$, originating from the Chern-Simons term, then becames a NC parameter of the formulation. Quantization of the NC system leads to quantum mechanics with ordinary product replaced by the Moyal product. Various new aspects arising in classical and quantum mechanics on NC space were observed (see [13]–[20] and references therein). An apparent defect of the NC models is a lack of relativistic invariance, mainly due to the fact that the noncommutativity parameter is a constant matrix.

In this work we demonstrate that the noncommutativity does not violate necessarily the Lorentz invariance, but does lead to non standard realization of the Lorentz group, at least for the particle models. In the abovementioned approach, coordinates of the extended configuration space are ordinary (commuting) quantities. So, there is no principal difficulty to construct a relativistic invariant theory on the extended space. After that, taking into account the constraints present in the model, one obtains a theory in physical subspace, the latter being noncommutative and endowed automatically with the Poincare structure. In particular, *NC brackets of the variables are covariant under the Poincare transformations.*

Let us discuss how this scheme can be realized for a D-dimensional particle (see [8] for details). Chern-Simons term can be added to the first order action of the relativistic particle, without spoiling the reparametrization invariance. As a consequence, the model will contain the desired relativistic constraint $p^2 - m^2 = 0$. The problem is that the numerical matrix $\theta^{\mu\nu}$ does not respect the Lorentz invariance. To resolve the problem, let us consider a particle interacting with a new configuration-space variable $\theta^{\mu\nu}(\tau) = -\theta^{\nu\mu}(\tau)$, instead of the constant matrix. An appropriate action is

$$S_0 = \int d\tau \left[\dot{x}^\mu v_\mu - \frac{e}{2}\left(v^2 - m^2\right) + \frac{1}{\theta^{\alpha\beta}\theta_{\alpha\beta}} \dot{v}_\mu \theta^{\mu\nu} v_\nu \right], \tag{1.1}$$

Here $x^\mu(\tau), v^\mu(\tau), e(\tau), \theta^{\mu\nu}(\tau)$, are the configuration space variables of the model, $\eta^{\mu\nu} = (+, -, \ldots, -)$. Insertion of the term θ^2 in the denominator has the same meaning as for the eibein in the action of massless particle: $L = \frac{1}{2e}\dot{x}^2$. Technically, it rules out the degenerate gauge $e = 0$. The action is manifestly invariant under the Poincare transformations $\delta x^\mu = \omega^\mu{}_\nu x^\nu + a^\mu$, $\delta v^\mu = \omega^\mu{}_\nu v^\nu$, $\delta\theta^{\mu\nu} = \omega^{[\mu}{}_\alpha \theta^{\alpha\nu]}$. The crucial observation is that the action is invariant also under the following local symmetry (with the parameter $\epsilon_{\mu\nu}(\tau) = -\epsilon_{\nu\mu}(\tau)$)

$$\delta x^\mu = -\epsilon^{\mu\nu} v_\nu, \qquad \delta\theta_{\mu\nu} = -\left(\theta^{\alpha\beta}\theta_{\alpha\beta}\right)\epsilon_{\mu\nu} + 2\theta_{\mu\nu}\left(\theta^{\alpha\beta}\epsilon_{\alpha\beta}\right), \tag{1.2}$$

associated with the variable θ. The latter can be gauged out (the corresponding first class constraints will be discussed below), an admissible gauge is

$$\theta^{0i} = 0, \qquad \theta^{ij} = \text{const}, \qquad \theta^2 \equiv \theta_{ij}\theta_{ji} \neq 0. \tag{1.3}$$

The situation here is similar to the string theory, where the two-dimensional metric $g_{ab}(\sigma^c)$ can be gauged out completely (using the reparametrization symmetry): $g_{ab} = \eta_{ab}$. The noncommutativity parameter of the gauge fixed version is the numerical matrix θ^{ij}. The model is also subject to second class constraints $G^\mu \equiv p^\mu - v^\mu = 0, T^\mu \equiv \pi^\mu - \frac{1}{\theta^2}\theta^{\mu\nu}v_\nu = 0$, where p, π are conjugated momenta for x, v. Let us construct the corresponding Dirac bracket. After that, taking x, p as the physical sector variables, remaining variables can be omitted from consideration by using the constraints. The physical subspace turns out to be noncommutative, with the following Dirac brackets for the physical variables:

$$\{x^\mu, x^\nu\} = 2\theta^{-2}\theta^{\mu\nu}, \qquad \{x^\mu, p_\nu\} = \delta^\mu_\nu, \qquad \{p_\mu, p_\nu\} = 0, \tag{1.4}$$

where eq. (1.3) is implied. Dynamics of the resulting theory corresponds to the free particle, the hamiltonian equations of motion are $\dot{x}^\mu = ep^\mu, \dot{p}^\mu = 0$. They are accompanied by the remaining first class constraint $p^2 - m^2 = 0$. As it usually happens in a theory with local symmetries, Poincare invariance of the gauge fixed version is combination of the initial Poincare and local transformations which preserve the gauge (1.3)

$$\begin{aligned} \delta x^\mu &= \omega^\mu{}_\nu x^\nu - \epsilon^{\mu\nu}(\omega) v_\nu + a^\mu, \qquad \delta v^\mu = \omega^\mu{}_\nu v^\nu; \\ \epsilon^{\mu\nu}(\omega) &\equiv -\theta^{-2}\omega^{[\mu}{}_\rho \theta^{\rho\nu]}, \end{aligned} \tag{1.5}$$

where eq. (1.3) for $\theta^{\mu\nu}$ is implied. By construction, *the brackets (1.4) are covariant with respect to the Poincare transformations (1.5)* (note that $v = p$ on the constraint surface). In the interacting theory Poincare invariance will require deformation of the free NC algebra, see (3.8), (3.12).

JHEP03(2003)021

Of course, the free particle is not very interesting object (similarly to the nonrelativistic case, any traces of noncommutativity disappear when formulated in terms of canonical variables $\hat{x}^\mu \equiv x^\mu + \frac{1}{\theta^2}\theta^{\mu\nu}v_\nu$, see also (1.1)). A possibility to introduce some rather exotic interaction with electromagnetic (EM) field was discussed previously in [21]). Below we demonstrate that NC relativistic particle couples to an external EM field by means of the standard term: $A_\mu \dot{x}^\mu$.

2. Partially gauge fixed action of NC relativistic particle and interaction with the electromagnetic field

To construct action for NC particle on EM background in the manifestly Poincare invariant form one needs to ensure, besides the Poincare symmetry, the local ϵ-symmetry (1.2) and U(1) gauge symmetry (another technically nontrivial task is to preserve the first class character of the relativistic constraint $p^2 - m^2 = 0$). To simplify the problem, we consider here partially gauge fixed version of the model which arises after fixing the gauge (1.3) for ϵ-symmetry . In this case, one needs to ensure the Poincare and U(1) symmetry of the action, which is a more simple task.

Starting from manifestly Poincare invariant action of NC particle (1) one finds in the hamiltonian formulation the following constraints

$$\begin{aligned} G^\mu \equiv p^\mu - v^\mu = 0\,, \qquad T^\mu \equiv \pi^\mu - \frac{1}{\theta^2}\theta^{\mu\nu}v_\nu = 0\,, \\ v^2 - m^2 = 0\,, \qquad p_\theta^{\mu\nu} = 0\,, \qquad p_e = 0\,. \end{aligned} \tag{2.1}$$

Poisson brackets of the constraints are

$$\begin{aligned} \{G^\mu, G^\nu\} = 0\,, \qquad \{T^\mu, T^\nu\} = -\frac{2}{\theta^2}\theta^{\mu\nu}\,, \qquad \{G_\mu, T^\nu\} = -\delta_\mu^\nu\,, \\ \{v^2 - m^2, T^\mu\} = 2v^\mu\,, \qquad \{T_\mu, p_\theta^{\rho\sigma}\} = -\frac{1}{\theta^2}\delta_\mu^{[\rho}v^{\sigma]} + \frac{4}{\theta^4}(\theta v)_\mu\theta^{\rho\sigma}\,. \end{aligned} \tag{2.2}$$

Using eq. (2.2) one can separate the first class constraints related with $\theta^{\mu\nu}$ variable

$$P_\theta^{\rho\sigma} = p_\theta^{\rho\sigma} - G^\alpha\{T_\alpha, p_\theta^{\rho\sigma}\} = 0\,. \tag{2.3}$$

$P_\theta^{\rho\sigma}$ has vanishing brackets with other constraints and corresponds to the local symmetry (1.2). In the gauge (1.3) for the constraints (2.3) only the space components of the variable $\theta^{\mu\nu}$ survive. The corresponding partially gauge fixed version of (1) is, evidently

$$S_{gf} = \int d\tau \left[\dot{x}^\mu v_\mu - \frac{e}{2}\left(v^\mu v_\nu - m^2\right) - \frac{1}{\theta^2}\dot{v}_i\theta^{ij}v_j\right], \tag{2.4}$$

where θ_{ij} *is now the numerical matrix from (1.3).* Physical sector dynamics of the models (1.1) and (2.4) is the same, which can be confirmed by direct computations.

Let us discuss interaction with the EM background. In the gauge fixed formulation (2.4), the Poincare transformation (1.5) for the variable x^μ involves v^μ. For this reason the background field $A^\mu(x)$ is not a Poincare covariant quantity. One can take a field with general dependence on the configuration space variables: $A^\mu(x, v)$. With this choice, equations of motion will have corrections of order $O(\theta/\theta^2)$ as well as other deviations from commutative case involving $\partial A_\mu/\partial v^\nu$. Since it is not a theory which one expects as NC version of ordinary particle, we consider here another possibility. Namely, from eq. (1.5) one finds Poincare invariant subspace parametrised by the coordinates

$$\hat{x}^\mu \equiv x^\mu + \frac{1}{\theta^2}\theta^{\mu\nu}v_\nu\,, \tag{2.5}$$

which obey $\delta_\omega \hat{x}^\mu = \omega^\mu{}_\nu \hat{x}^\nu$. (It is interesting to note that they coincide with the canonical coordinates of the free theory). The field $A^\mu(\hat{x})$ is a Poincare covariant object. So, as the interaction theory, let us consider the following action

$$S = \int d\tau \left[\dot{x}^\mu v_\mu - \frac{e}{2}\left(v^\mu v_\mu - m^2\right) - \frac{1}{\theta^2}\dot{v}_i\theta^{ij}v_j + \dot{x}^\mu A_\mu\left(x + \frac{\theta v}{\theta^2}\right)\right]. \tag{2.6}$$

Note that θ-dependence can not be removed now by a change of variables, similarly to the case of NC quantum mechanics with a potential. Symmetries of the action are reparametrizations, Poincare symmetry and local gauge symmetry associated with $A^\mu(\hat{x})$. The Poincare transformations are represented by eq. (1.5), accompanied by $\delta A^\mu = \omega^\mu{}_\nu A^\nu - n_\mu(\dot{v}\epsilon(\omega)A)$. Here $n^\mu \equiv \dot{x}^\mu/(\dot{x})^2$ is a unit vector in the direction of motion of the particle.[1] The transformation is non singular on-shell.

Since the background field depends both on x^μ and v^μ, one expects that U(1) symmetry of the gauge fixed version will have a corrected form (note that the symmetry $\delta A_\mu = \partial_\mu\alpha(x)$ is not sufficient to gauge away the time-like component $A_0(\hat{x})$). It is indeed the case, and local gauge symmetry of (2.6) has the form

$$\delta A_\mu(\hat{x}) = \partial_\mu\alpha(\hat{x}) - \theta^{-2}n_\mu\left(\dot{v}\theta\partial\alpha(\hat{x})\right), \tag{2.7}$$

with the parameter $\alpha(\hat{x})$. Here and below, the derivative ∂_μ is calculated with respect to $\hat{x}^\mu$. As in the commutative space, the symmetry (2.7) ensures absence of negative norm modes of EM field.

3. Hamiltonian analysis of the model

To analyze physical sector of the constrained system (2.6), we rewrite it in the hamiltonian form. The interaction term leads to a nonlinear deformation of the constraint structure as compared with the free case. In all the equations below $\theta^{\mu\nu}$ is the numerical matrix (1.3). The following notations will be used:

$$\hat{\theta}^{\mu\nu} \equiv 2\theta^{-2}\theta^{\mu\nu}\,, \qquad \triangle_{\mu\nu} \equiv \eta_{\mu\nu} - \theta^{-2}\theta_\mu{}^\alpha\partial_\alpha A_\nu\,,$$
$$D \equiv \left[\triangle + \hat{\theta}(\triangle^T)^{-1}F\right]^{-1}, \tag{3.1}$$

[1] Appearance of velocity dependent terms in the Poincare transformation law is a common property of gauge fixed versions. For example, the transformations for ordinary particle in the light-cone gauge are of the form $\delta x^\mu = \omega^\mu{}_\nu x^\nu - (\omega^0{}_\nu x^\nu)\dot{x}^\mu$, $\delta v^\mu = \omega^\mu{}_\nu v^\nu - (\omega^0{}_\nu x^\nu)\dot{v}^\mu$, $\delta A^\mu = \omega^\mu{}_\nu A^\nu - (\omega^0{}_\alpha x^\alpha)\dot{x}^\nu\partial_\nu A_\mu$.

where $F_{\mu\nu}$ is the field strength. Starting from the action (2.6), one finds in the hamiltonian formalism the primary constraints

$$G^\mu \equiv p^\mu - v^\mu - A_\mu(\hat{x}) = 0\,, \qquad T^\mu \equiv \pi^\mu + \frac{1}{\theta^2}\theta^{\mu\nu}v_\nu = 0\,, \\ p_e = 0\,. \tag{3.2}$$

and the hamiltonian

$$H = \frac{e}{2}\left(v^\mu v_\mu - m^2\right) + \lambda_{1\mu}G^\mu + \lambda_{2\mu}T^\mu + \lambda_e p_e\,. \tag{3.3}$$

Here p, π are conjugated momentum for x, v and λ are the lagrangian multipliers for the constraints. On the next step there is appear the secondary constraint

$$v^\mu v_\mu - m^2 = 0\,, \tag{3.4}$$

as well as equations for determining the lagrangian multipliers

$$\lambda_1 = eDv\,, \qquad \lambda_2 = e(\triangle^T)^{-1}FDv\,. \tag{3.5}$$

There is no of tertiary constraints in the problem. Poisson brackets of the constraints are

$$\{G^\mu, G^\nu\} = F^{\mu\nu}\,, \qquad \{T^\mu, T^\nu\} = \frac{2}{\theta^2}\theta^{\mu\nu}\,, \qquad \{T_\mu, G_\nu\} = \triangle_{\mu\nu}\,. \tag{3.6}$$

The constraints G^μ, T^μ form the second class system and can be taken into account by transition to the Dirac bracket. Its manifest form is

$$\begin{aligned}\{A,B\}_D = {} & \{A,B\} - \{A,G_\mu\}\left[D\hat{\theta}\left(\triangle^T\right)^{-1}\right]^{\mu\nu}\{G_\nu,B\} - \{A,G_\mu\}D^{\mu\nu}\{T_\nu,B\} + \\ & + \{B,G_\mu\}D^{\mu\nu}\{T_\nu,A\} - \{A,T_\mu\}\left[\left(\triangle^T\right)^{-1}FD\right]^{\mu\nu}\{T_\nu,B\}\,.\end{aligned} \tag{3.7}$$

Now the constraints (3.2) can be used to omit part of variables from consideration. Since the constraint $G^\mu = 0$ can not be resolved with respect to v^μ, we take the variables (x, v) as the physical sector ones.[2] The Dirac brackets (in the matrix notations) for the physical variables are

$$\begin{aligned}\{x,x\} &= D\hat{\theta}(\triangle^T)^{-1}\,, \qquad \{x,v\} = D\,, \\ \{v,v\} &= (\triangle^T)^{-1}FD\,.\end{aligned} \tag{3.8}$$

In the decoupling limit $A^\mu \to 0$ one has the free particle NC algebra (1.4). The hamiltonian equations of motion for the theory are

$$\dot{x} = eDv\,, \qquad \dot{v} = e\left(\triangle^T\right)^{-1}FDv\,. \tag{3.9}$$

[2]First order formulation implies that one can choose different pairs of variables as the physical ones [5]. It leads to equivalent descriptions of the system, as it should be. For example, for ordinary particle on EM background the sectors x, p and x, v are related as follows: $p^\mu = v^\mu + A^\mu$.

They are accompanied by the constraint $v^2 - m^2 = 0$. The equations (3.9) imply the following second order equation for the coordinate of the particle (in the gauge $e = 1$):

$$\ddot{x} = D\left[\left(\triangle^T\right)^{-1} F - \left(D^{-1}\right)^{\cdot}\right]\dot{x}\,. \tag{3.10}$$

Poincare invariant dynamics on NC space acquires nontrivial corrections of order $\hat{\theta}$, as compared with the ordinary particle. Expanding r.h.s. of eqs. (3.8)–(3.10) up to $O^2(\hat{\theta})$, one obtains the first order corrections as follows:

$$\begin{aligned}
\dot{x}^\mu &= ev^\mu - e\left[\hat{\theta}\left(F - \frac{1}{2}\partial A\right)\right]^{\mu\nu} v_\nu\,, \\
\dot{v}^\mu &= eF_{\mu\nu}v^\nu + e\left[\frac{1}{2}\left(\hat{\theta}v\right)^\alpha \partial_\alpha F_{\mu\nu} - \left(F\hat{\theta}F\right)_{\mu\nu} + \frac{1}{2}\left(F\hat{\theta}\partial A\right)_{[\mu\nu]}\right] v^\nu\,.
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
\{x^\mu, x^\nu\} &= \hat{\theta}^{\mu\nu}, \\
\{x^\mu, v_\nu\} &= \delta^\mu_\nu - \hat{\theta}^{\mu\alpha}\left(F - \frac{1}{2}\partial A\right)_{\alpha\nu}, \\
\{v_\mu, v_\nu\} &= F_{\mu\nu} + \frac{1}{2}\left(\hat{\theta}v\right)^\alpha \partial_\alpha F_{\mu\nu} - \left(F\hat{\theta}F\right)_{\mu\nu} + \frac{1}{2}\left(F\hat{\theta}\partial A\right)_{[\mu\nu]}\,.
\end{aligned} \tag{3.12}$$

$$\ddot{x} = F\dot{x} + \frac{1}{2}\left(\hat{\theta}v\right)^\alpha \partial_\alpha F\dot{x} - \hat{\theta}FF\dot{x} + \frac{1}{2}\hat{\theta}\partial AF\dot{x} - \frac{1}{2}\dot{x}F\hat{\theta}\partial A - \hat{\theta}\left(F - \frac{1}{2}\partial A\right)^{\cdot}\dot{x}. \tag{3.13}$$

In the commutative limit $\hat{\theta} \to 0$ the equations (3.11)–(3.13) turn out into the corresponding equations for ordinary particle on the background $A^\mu(x)$. In this limit the Poincare transformations acquire the standard form (the variable v^μ disappears from the transformation law for x^μ).

4. Conclusion

In this work we have presented D-dimensional NC relativistic particle coupled to the background EM field. To construct Poincare invariant theory on NC space one needs to replace the numerical NC parameter $\theta^{\mu\nu}$ by a dynamical variable. Local ϵ-symmetry associated with the variable allows one to impose the gauge $\theta^{0i} = 0, \theta^{ij} = \text{const}$. The numerical matrix θ^{ij} turns out to be a NC parameter of the physical subspace. The standard interaction term can be added to the partially gauge fixed action for the free NC particle (2.4) without spoiling the Poincare symmetry or the U(1) gauge symmetry. Poincare invariance on the NC space requires special dependence of the background field on the variables of the first order formulation: $A^\mu(x + \frac{\theta v}{\theta^2})$ (for our choice of the physical variables, they turn out into phase space variables in the hamiltonian formulation). In turn, this implies modified form of the U(1)-transformations (2.7). Another consequence of this special dependence is that the Poincare invariant dynamics on NC space acquires higher nonlinear corrections of order $\hat{\theta}$ as compared with ordinary particle. Let us point that the method developed can be directly applied to investigate the rotational invariance in the NC quantum mechanics as well.

JHEP03(2003)021

The equations (3.11)–(3.13) can be a starting point for various computations of leading corrections to ordinary (commutative) theory. An important conclusion from eq. (3.12) is that in an interacting Poincare invariant theory one can not use the free particle NC algebra (1.4) as the leading approximation in $\hat{\theta}$, since the latter is given by (3.12). Nonrelativistic limit of the algebra (3.12) gives the corresponding NC quantum mechanics with brackets which are different from (1.4). It can imply revision of the results obtained in NC quantum mechanics on the basis of eq. (1.4) [17, 19, 20]. To analyze these problems, it will be interesting to find a set of canonical variables (i.e. the ones with canonical brackets instead of eq. (3.12). Another interesting problem is to find a manifestly Poincare invariant version of the model (2.6). We suppose also that that the scheme can be realised for the open string on a B-field background. The formulation developed in [6] in terms of infinitely dimensional mechanics seems to be appropriate for this aim. These problems will be discussed elsewhere.

References

[1] M. Kontsevich, *Deformation quantization of Poisson manifolds, I*, q-alg/9709040.

[2] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[3] G. Dunne and R. Jackiw, *"Peireles substitution" and Chern-Simons quantum mechanics*, *Nucl. Phys.* **33C** *(Proc. Suppl.)* (1993) 114.

[4] J. Lukierski, P.C. Stichel and W.J. Zakrzewski, *Galilean-invariant* $(2+1)$*-dimensional models with a Chern-Simons-like term and* $D = 2$ *noncommutative geometry*, *Ann. Phys. (NY)* **260** (1997) 224 [hep-th/9612017]; *Noncommutative planar particle dynamics with gauge interactions*, hep-th/0207149.

[5] A.A. Deriglazov, *Quantum mechanics on noncommutative plane and sphere from constrained systems*, *Phys. Lett.* **B 530** (2002) 235.

[6] A.A. Deriglazov, C. Neves and W. Oliveira, *Open string with a background b-field as the first order mechanics and noncommutativity*, hep-th/0110183.

[7] A.A. Deriglazov, *Noncommutative version of an arbitrary nondegenerated mechanics*, hep-th/0208072.

[8] A.A. Deriglazov, *Noncommutative relativistic particle*, hep-th/0207274.

[9] P.A.M. Dirac, *Lectures on quantum mechanics*, Yeshiva University, NY 1964.

[10] D.M. Gitman and I.V. Tyutin, *Quantization of fields with constraints*, Springer-Verlag, Berlin 1990.

[11] L.D. Faddeev and R. Jackiw, *Hamiltonian reduction of unconstrained and constrained systems*, *Phys. Rev. Lett.* **60** (1988) 1692.

[12] Z. Guralnik, R. Jackiw, S.Y. Pi and A.P. Polychronakos, *Testing non-commutative qed, constructing non-commutative mhd*, *Phys. Lett.* **B 517** (2001) 450 [hep-th/0106044].

[13] L. Susskind, *The quantum hall fluid and non-commutative Chern-Simons theory*, hep-th/0101029.

JHEP03(2003)021

[14] M.M. Sheikh-Jabbari, *Discrete symmetries (C,P,T) in noncommutative field theories*, *Phys. Rev. Lett.* **84** (2000) 5265 [hep-th/0001167].

[15] M. Chaichian, M.M. Sheikh-Jabbari and A. Tureanu, *Hydrogen atom spectrum and the lamb shift in noncommutative QED*, *Phys. Rev. Lett.* **86** (2001) 2716 [hep-th/0010175].

[16] V.P. Nair and A.P. Polychronakos, *On level quantization for the noncommutative Chern-Simons theory*, *Phys. Rev. Lett.* **87** (2001) 030403.

[17] A. Mazumdar and M.M. Sheikh-Jabbari, *Noncommutativity in space and primordial magnetic field*, *Phys. Rev. Lett.* **87** (2001) 011301.

[18] S. Bellucci and A. Nersessian, *Phases in noncommutative quantum mechanics on (pseudo)sphere*, *Phys. Lett.* **B 542** (2002) 295 [hep-th/0205024].

[19] K. Bolonek and P. Kosinski, *On uncertainty relations in noncommutative quantum mechanics*, *Phys. Lett.* **B 547** (2002) 51 [hep-th/0208162];
P. Kosinski and K. Bolonek, *Minimalisation of uncertainty relations in noncommutative quantum mechanics*, hep-th/0208213.

[20] L. Jonke and S. Meljanac, *Representations of noncommutative quantum mechanics and symmetries*, hep-th/0210042.

[21] A.A. Deriglazov, *Noncommutative relativistic particle on the electromagnetic background*, *Phys. Lett.* **B 555** (2003) 83 [hep-th/0208201].